Quantum Counting with Decoherence Errors
— Influence of Circuits' Order —

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Abstract. Quantum counting algorithm can estimate the number of solutions in a database. In this paper, we analyze its behaviors in the presence of decoherence errors. We construct two counting circuits in which the order of commutative gates are altered and show that the behaviors of circuits are different when decoherence errors occur. On one circuit, almost correct number of solutions are obtained and on the other circuit, the estimated numbers of solutions are mostly distributed over the correct one and wrong counts zero and N, the number of elements. We analyze that these phenomena result from the rotation outside the Grover space due to decoherence.

Keywords: quantum counting, decoherence errors, Grover space

1 Introduction

At quantum computations, it is necessary to perform unitary transform with maintaining coherency. There exist, however, decoherence errors inevitably so that decoherence of the state occurs. Therefore, it is important to analyze the behaviors of quantum algorithms with decoherence.

Quantum counting algorithm [1] can estimate the number of solutions in a search space quadratically faster than classical algorithms by rotating states in the Grover space. This algorithm is constructed by utilizing Grover's database search algorithm [2, 3] and quantum Fourier transform (QFT). However, when decoherence errors occur, quantum states move out of the Grover space. Hence, this rotation can not be represented in the Grover space, and this behavior has been unknown to the best of our knowledge.

In this paper, we analyze a behavior of quantum counting algorithm in the presence of decoherence errors. In order to consider the action of this rotation outside the Grover space, we examine the action on an arbitrary initial state and show that two wrong peaks at 0 and N frequency result from this rotation, where N is the number of elements. Moreover, in order to analyze the peaks of probability distribution, we construct two quantum counting circuits with altering the order of commutative gates. Behaviors are different and we show that this reason results from the action of rotation outside the Grover space. From our results, we can refer to the importance of the order of quantum gates for constructing the effective quantum circuits and analyzing the behaviors of circuits exactly.

2 Preliminaries

2.1 Grover's database search algorithm

Grover's database search algorithm [2, 3] searches "good" element, or "solution", in unordered N elements with $O(\sqrt{N})$ operations. Suppose there are t solutions and $N = 2^n$. This algorithm consists of repeated application of Grover iteration G on *Grover space* with the basis $\{|g\rangle, |b\rangle\}$ defined as follows. Let the states of solutions be $|g_1\rangle, \ldots, |g_t\rangle$ and non-solutions $|b_{t+1}\rangle, \ldots, |b_N\rangle$. We divide the Hilbert space $\mathcal{H}$ into the direct sum of two subspaces $\mathcal{H}_g$ and $\mathcal{H}_b$.

$$\mathcal{H}_g = |g_1\rangle \oplus \cdots \oplus |g_t\rangle,\ \mathcal{H}_b = |b_{t+1}\rangle \oplus \cdots \oplus |b_N\rangle. \quad (1)$$

Using these Hilbert spaces, let the states $|g\rangle$ ($|b\rangle$) be the superposition of solutions (non-solutions) to the search program:

$$|g\rangle = \frac{1}{\sqrt{t}} \sum_{\text{good}} |x\rangle \in \mathcal{H}_g,\ |b\rangle = \frac{1}{\sqrt{N-t}} \sum_{\text{bad}} |x\rangle \in \mathcal{H}_b, \quad (2)$$

where "good" indicates the set of solutions and "bad" non-solutions. The Grover space is spanned by $|g\rangle$ and $|b\rangle$.

On this space, Grover operator G consists of two steps:

1. Apply $U_1 : |g\rangle \mapsto -|g\rangle, |b\rangle \mapsto |b\rangle$
2. Apply $U_2 := (2|s\rangle\langle s| - I)$ where $|s\rangle := \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$.

The operator U_1 flips the sign of the good states, and is called Oracle operator. The operator U_2 flips against the average $|s\rangle$. Then, Grover operator $G = U_2 U_1$. G is also represented as $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ in the Grover space with the basis $\{|g\rangle, |b\rangle\}$, where $\sin(\theta/2) = \sqrt{t/N}$.

2.2 Quantum counting algorithm

Brassard et al. [1] devised the fast quantum algorithm to count the number of solutions t in N elements. The quantum counting algorithm is based on the Grover's algorithm and QFT. The Grover's algorithm has a period related to θ determined by t, and QFT estimates this period.

On the quantum counting circuit in Figure 1, G is applied to $n + 1$ qubits and θ is estimated by using p qubits. Let the result of measurement be $\tilde{\varphi}$. The angle of iteration G equals to $\tilde{\theta} = (\tilde{\varphi}/P)2\pi$, so that the number of solutions equals to $\tilde{t} = N\sin^2(\tilde{\theta}/2) = N\sin^2(\tilde{\varphi}/P)\pi$.

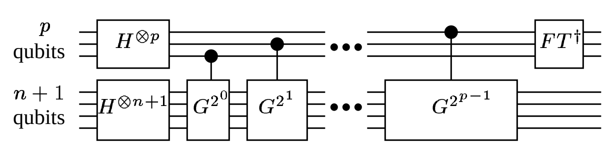

Figure 1: Quantum counting circuit (a)

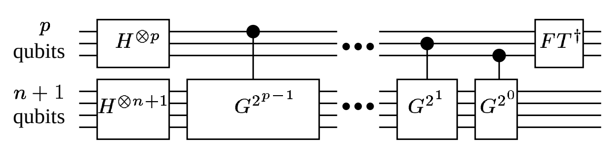

Figure 2: Quantum counting circuit (b)

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2.3 Decoherence errors

We consider the depolarizing channel as the decoherence error model [4]: at each step, we apply Pauli gate σ_x, σ_y or σ_z to each qubit independently with probability $p/3$, or leave it unchanged. The influence of decoherence errors depend on the product of the depth and the number of qubits of the circuit [5].

3 Arbitrary initial state

Quantum counting estimates a phase of Grover iteration G. However, when decoherence errors occur, the states move out of the Grover space. We examine the action of G outside the Grover space by considering the arbitrary initial state.

For simplicity, we limit ourselves to the case dim $\mathcal{H}_g \geq 2$ and dim $\mathcal{H}_b \geq 2$. Let an arbitrary initial state:

$$|\psi\rangle = u|g\rangle + u'|e_g\rangle + v|b\rangle + v'|e_b\rangle, \qquad (3)$$

where $|e_g\rangle \in \mathcal{H}_g$ $(|e_b\rangle \in \mathcal{H}_b)$ is the unequal superposition of good (bad) states which satisfies $\langle g|e_g\rangle = \langle b|e_b\rangle = 0$, $\langle e_g|e_g\rangle = \langle e_b|e_b\rangle = 1$. $|e_g\rangle$ and $|e_b\rangle$ are uniquely defined due to Gram-Schmidt orthogonalization. The oracle operator U_1 acts as

$$U_1 : |g\rangle \mapsto -|g\rangle, |e_g\rangle \mapsto -|e_g\rangle, |b\rangle \mapsto |b\rangle, |e_b\rangle \mapsto |e_b\rangle, \qquad (4)$$

since U_1 only flips the states belonging to $\mathcal{H}_g$.

We apply Grover iteration $G = U_2 U_1$ to the initial state (3):

$$|\psi\rangle \stackrel{G}{\mapsto} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \oplus u' \oplus -v'. \qquad (5)$$

Therefore, this vector space can be decomposed into three subspaces $\{|g\rangle, |b\rangle\}$, $\{|e_g\rangle\}$ and $\{|e_b\rangle\}$ under the operator G.

Let the angle obtained on quantum counting be $\tilde{\theta}$. From above rotation (5), each angle $\tilde{\theta}$ of the rotation G equals to $\theta, 0$ or π in each subspace $\{|g\rangle, |b\rangle\}$, $\{|e_g\rangle\}$ or $\{|e_b\rangle\}$, respectively. Therefore, according to $\tilde{t} = N\sin^2(\tilde{\theta}/2)$, the number of solutions $\tilde{t} = t$, 0 and N are obtained with high probability in the presence of decoherence errors.

4 Peaks on quantum counting

We constructed two quantum counting circuits in Figure 1, 2. On two circuits, the order of quantum gates G are different. These gates are commutative with no error, so that the behaviors of two circuits are exactly the same in this case.

4.1 Probability amplitude on two circuits

We did experiments on Quantum Computation Simulation System (QCSS) [4] with 10000 trials. Figure 3 and 4 show the results of our simulation on quantum counting to count the number of solutions satisfying $x \leq 12$, where x is in the range $[0, 63]$. Figure 3 and 4 represent the probability that the number of solutions $\tilde{t}$ is obtained with decoherence error rate $d = 10^{-4}$ on Circuit (a) and Circuit (b), respectively. As shown in these figures, the behaviors of two circuits are different. On Circuit (a), counts near correct one 13 are obtained. On the other hand, on Circuit (b), wrong counts 0 and $N = 64$ are obtained with high probability. Furthermore, the correct number of solutions are obtained with higher probability on Circuit (a) than on Circuit (b). We clarify the phenomena.

4.2 Discussion of the reason for the peaks

Assume the decoherence occurs in lower half of Circuit (b) where G is applied. Then, states turn out of the Grover space, e.g., $|e_g\rangle$ or $|e_b\rangle$ space. Notice that the original quantum counting processes contain such part as $\sum_{lm}|l\rangle|m\rangle \mapsto \sum_{lm}|l\rangle G^l|m\rangle$. Let $l := \sum_{i=0}^{p-1} l_i 2^i (l_i = \{0,1\})$, then

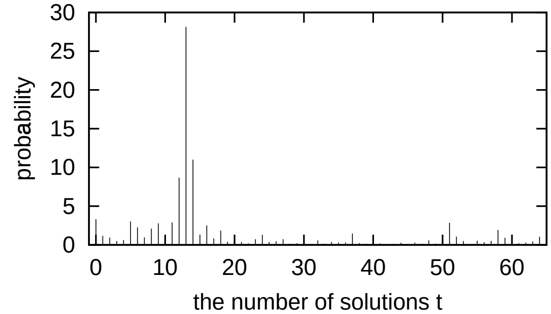

Figure 3: Probability with $d = 10^{-4}$ on Circuit (a)

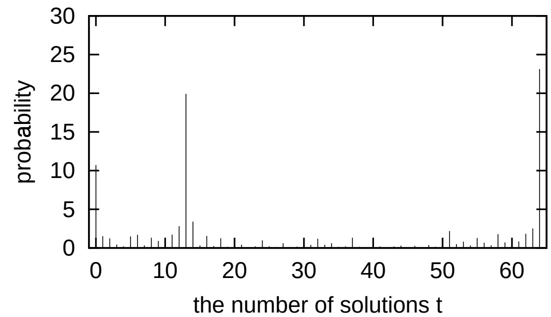

Figure 4: Probability with $d = 10^{-4}$ on Circuit (b)

$G^l = G^{l_0} G^{2l_1} \cdots G^{2^{p-1}l_{p-1}}$. Each $G^{2^j l_j}$ corresponds to controlled unitary gate in Figure 2 and is arranged in turn. Next, consider the case when one error gate D is sandwiched within these controlled gate sequence:

$$G^{l_0} G^{2l_1} \cdots G^{2^{k-1}l_{k-1}} D G^{2^k l_k} \cdots G^{2^{p-1}l_{p-1}}. \qquad (6)$$

In our model, decoherence is treated as depolarizing channel. Hence, we can assume $D = I \otimes \cdots \otimes I \otimes \sigma_k \otimes I \otimes \cdots \otimes I (k = x, y, z)$. Then the expression (6) can be represented as

$$D G'^{l_0} G'^{2l_1} \cdots G'^{2^{k-1}l_{k-1}} G^{2^k l_k} \cdots G^{2^{p-1}l_{p-1}}, \qquad (7)$$

where $G' := D^{-1}GD$. Namely, every controlled-G after the error gate D is modified to the controlled-G'. As a result, the quantum counting circuit with decoherence errors estimates the phases which correspond not only to the rotation of G but also that of G'. When G' is applied, the wrong counts 0 and N are frequently estimated from Section 3.

The probability that D is applied first after G^{2^k} is $(2^k - 1)/(2^p - 1)$. Thus, this probability becomes high when k is near p. This means that bits of the measurement result except the upper ones are changed into another bits associated with 0 and N from Figure 2.

Similarly, on Circuit (a), the lower bits of measurement result are changed into another bits. Thus, the counts near correct one are obtained with high probability. In addition, there are measurement results near correct one that $\tilde{t}$ obtained by calculating these results equals to the correct counts t, so that correct count is obtained with higher probability on Circuit (a) than on Circuit (b). As a result, Circuit (a) is more robust against decoherence errors than Circuit (b).

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