Quantum Cloning Under Decoherence and An Application of Cloning

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Abstract

Quantum cloning is an interesting topic in quantum information, as they exemplify the significant difference between classical and quantum mechanics and are hoped to be a useful element in quantum information processing. Here we examine how many clones can be generated when the system is fraught with the problem of decoherence. We first find a method to construct an $N \to M$ cloning circuit, and compare the cloning time and the decoherence time. It turns out that our circuit is highly vulnerable to decoherence when it is implemented with ion traps. We also have studied the application of quantum cloning to error-correction.

Keywords: Quantum cloning, decoherence, ion traps, error-correction.

I. INTRODUCTION

Despite the impossibility of making perfect copies of an unknown quantum state (no-cloning theorem), the theory of quantum mechanics allows us to clone quantum states approximately. Since the discovery of the *universal quantum cloning machine (UQCM)* by Bužek and Hillery [1], many theoretical works have been carried out and some experiments to copy photons have been also performed [2]. We have investigated the effect of decoherence on the quantum cloning by considering a general configuration of its network and estimated the upper bound for the number of cloneable states which can be generated in a decoherent environment. Most recently, we have studied the application of quantum cloning to error-correction.

II. GENERIC QUANTUM CLONER AND CIRCUIT COMPLEXITY

Bužek et al. also presented a way to construct a quantum circuit for $1 \to M$ as well as $1 \to 2$ UQCMs [3]. In both cases, the whole process is divided into two stages, namely preparation and cloning stages.

How can we construct more generic $N \to M$ quantum cloning circuits? A natural generalisation of the circuit of $1 \to M$ UQCM would look like Figure 1.

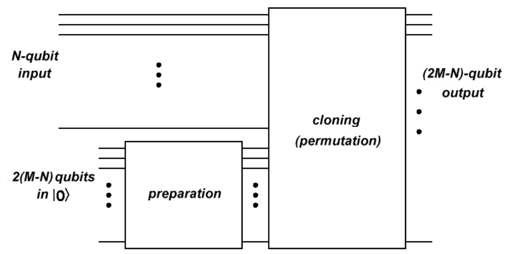

FIG. 1: Quantum circuit for $N \to M$ UQCM. It turns out that this cloning circuit does work properly for any M when $N = 1$, $N = 2$, and many other N and M as long as the condition (1) is fulfilled.

Since the task of the cloning stage, consisting of a sequence of CNOT gates [3], is to permute the amplitudes of 2^{2M-N}-dimensional Hilbert space, all the amplitudes in the $N \to M$ cloning transformation [4] must be generated by the preparation stage, although the number of distinct amplitudes is only $M - N + 1$. This imposes a condition upon N and M in order for an $N \to M$ UQCM to work, and it turns out to be

$$\sum_{k=0}^{M-N} \binom{M}{k} \binom{M-N}{k} \leq 2^{2(M-N)}. \qquad (1)$$

The LHS of Eq. (1) represents the number of bases that appear in the transformation and the RHS is the dimension of the Hilbert space where qubits in the preparation stage lie. This condition cannot necessarily be always satisfied, nevertheless, it is satisfied if M is sufficiently large compared with N.

The cloning stage can be built as follows. As mentioned above, the cloning stage's task is to permute the bases to distribute amplitudes generated by the preparation stage to proper destinations. Thus, we can make use of $\mathrm{C}^k\mathrm{NOT}^l$ gate, which flips l target bits depending on the values of k control bits. Any permutation can be carried out by a sequence of such gates.

With the configuration described above, an $N \to M$ UQCM circuit will have $O(2^{2M}(M-N)^2(2^{-2N}+M^{-\frac{1}{2}}))$ CNOTs at most in total and we will take this value as the circuit complexity to estimate the effect of decoherence in a physical implementation in the following.

III. ION TRAPS AND THE NUMBER OF CLONEABLE QUBITS

With the number of CNOT gates estimated above, we can compare the cloning time T with the decoherence time τ_{dec}. We take cooled trapped ions (Ca^+, Hg^+ and Ba^+) and spontaneous emission as an example of possible implementations and decoherence.

Since the processing time for each CNOT is proportional to the inverse of the Rabi frequency between the levels in consideration ($|0\rangle$ and $|1\rangle$), we may wish to increase the intensity of the driving laser to minimise T. However, a stronger laser field would cause transitions to higher levels, which may decay faster by spontaneous emissions. Thus, we need to minimise the probability of spontaneous emission from the level $|1\rangle$ and higher levels.

By minimising the probability of spontaneous emission, i.e., the decoherence time, with respect to the electric field intensity, we have estimated the upper bounds for the number of clonal qubits. As a result, we see that even for a small number of outputs copied from one input, the decoherence due to spontaneous emission plays a critical role. Even with an optimistic assumption for the Lamb-Dicke parameter, $\eta = 1.0$, $1 \to 2$ cloning with Ca^+, $1 \to 2$ and $2 \to 3$ with Ba^+ would be possible [5].

IV. ERROR-CORRECTION WITH CLONING CIRCUIT

We turn to our attempt to apply quantum cloning to error-correction. Our aim is to reduce the redundancy when correcting errors that occur during transmission through a noisy channel.

The key point in this (approximate) error-correcting scheme is to acquire some information on the incoming state by measuring two of the output qubits from the cloning circuit. Probability distributions for outcomes from two qubits tell us the tendency concerning α, β and ϕ, when the incoming state can be written as $|\psi\rangle = \alpha|0\rangle + \beta e^{i\phi}|1\rangle$.

The protocol goes as follows. Alice measures two of three qubits emerging from the cloning circuit and sends this two-bit information to Bob classically. Bob compares Alice's outcomes with his own measurement outcomes. Since each outcome implies which "quadrant" of the $\alpha - \phi$ plane the state lies in, Bob can infer which of bit and phase flips, or both, happened to the state in the channel. If these outcomes disagree, Bob flips the bit or the phase or both, according to the discrepancy.

In order to improve the final fidelity as much as possible, we also attempt to reverse the quantum measurement, approximately and deterministically. Each reversal matrix corresponding to each measurement outcome i turns out to be a unitary matrix U_i, which appears in the polar decomposition of E_i, one of the Kraus operators for quantum cloning, i.e., $U_i = E_i \left(\sqrt{E_i^\dagger E_i}\right)^{-1}$.

The average of fidelity between initial and final states over the $\alpha - \phi$ plane is 0.592, which is rather low if we see it as an error-correcting protocol. An even simpler method involving a direct measurement and the reproduction of the state can give a better average, 2/3. Nevertheless, this fidelity is, interestingly, independent of the error probabilities of the channel.

V. REMARKS

We have investigated a possible method to construct an $N \to M$ UQCM circuit. With the circuit complexity we obtained for this circuit, it has been shown that quantum cloning may be vulnerable to decoherence. We have further studied quantum cloning as a potential "tool" in quantum communication, introducing the reversal operation. The overall efficiency of our protocol as an error-correcting scheme is rather low. Even though this result is somewhat negative, we hope that this will give an implication on the manipulation of information, for example, the separation of classical and quantum information.

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